d.) Fluctuations in Plasma and the Test Particle Model

Contents

i.) Basic Ideas of Fluctuation and Dissipation in Low Collisionality Plasma

ii.) Test Particle Model of Thermal Equilibrium Spectrum
   a.) → Calculation
   b.) → key assumptions

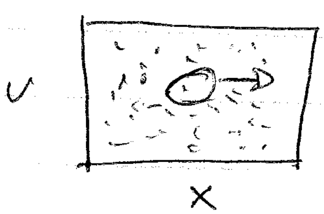
iii.) Weakly Non-Equilibrium Relaxation ↔ Lenard-Balescu Equation

iv.) Dynamic Relaxation

(i) Basic Ideas – Equilibrium Fluctuations

→ plasma:
- $1/n\lambda_D^3 \ll 1$ ⇒ many particles in Debye sphere
- $k_B T \gg e^2/r$ ⇒ thermal energy dominates electrostatic energy

→ Equilibrium: Balances
→ absorption vs. emission
(equivalent) → fluctuation vs. dissipation

i.e.  ① emission: discrete particle in Vlasov fluid emits waves?

$$\underline{\nabla}\cdot\underline{D} = \underline{\nabla}\cdot\underline{\underline{\epsilon}}\,\underline{E} = 4\pi n_0 q\,\delta(\underline{x}-x(t))$$

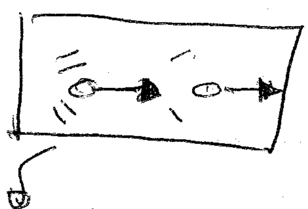
i.e. → a la' boat wake on water
→ Cerenkov emission

∴ discrete emission ⇔ fluctuation

i.e. – particle kinetic energy coupled to wave energy
– Cerenkov emission ⇔ particle slowing down

② Cerenkov emitted waves damp via
→ Landau damping
→ in/on Vlasov fluid (i.e. Landau damping for $\nu \to 0$)

i.e.



emitted waves damp → $\left\{ \begin{array}{l} \text{absorption} \\ \text{dissipation} \end{array} \right\}$ heats particles

So, conceptual picture of thermal equilibrium fluctuations is detailed balance of:

→ Cerenkov emission of waves from individual discrete particles

→ absorption of waves via Landau damping on Vlasov fluid

NB: ① Here, assume periodic B.C.'s → no radiative damping, outgoing waves, etc.

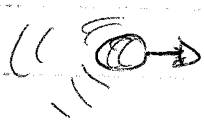
Note in this picture, each particle plays a dual role (i.e. "double agent"):

② In general, take damping length finite

i.e. $l_{damp} \sim \frac{\omega}{k} / |\gamma_k| < L_{system}$

As:

- "emitter": a discrete particle moving along some specified (unperturbed) orbit

i.e.  an identifiable 'pea' in a 'pea soup' composed of other peas.

- "absorber": an element of the Vlasov fluid responding to, and Landau damping emission from (other) discrete particles

i.e.  a "crushed pea" element of the "pea soup" of the Vlasov fluid.

so

→ equilibrium plasma = soup/gas of 'dressed' test particles

↑

Vlasov fluid screening

⇒ "test particle model"

→ every pea in the soup acts like soup for all the other peas ....

Note: Useful Analogy

	Brownian Motion	Equilibrium Plasma
Fluctuation	$\langle \tilde{v}^2 \rangle_\omega$	$\langle \tilde{E}^2 \rangle_{k,\omega}$
Exciton	ω-mode	$k,\ \omega(k) \rightarrow$ plasma waves
Emission/Source	$\langle \tilde{f}^2 \rangle$ – fluid thermal force	particle discreteness
Absorption/Damping	$\beta \leftrightarrow$ Stokes Drag on Particle	$\epsilon_{IM} \leftrightarrow$ Landau Damping of Collective Modes

n.b.: "Brownian Motion"

$$\frac{d\tilde{\underline{v}}}{dt} + \beta \tilde{\underline{v}} = \frac{\tilde{\underline{f}}}{m}$$

ii) Test Particle Model – Fluctuation Spectrum

→ As noted before, basic idea is that:

- each particle both a 'discrete emitter' and participant in Vlasov fluid screening cloud
- fluctuations weak → unperturbed orbits valid.

∴ if consider stationary cons:

$$\delta f = \underset{\substack{\downarrow \\ \text{coherent} \\ \text{Vlasov} \\ \text{response} \\ \text{(screening)}}}{f^c} + \underset{\substack{\downarrow \\ \text{discrete} \\ \text{particle source}}}{\tilde{f}}$$

(cf. Debye calculation)

$$\delta f = \underbrace{\frac{\frac{|e|}{m} \tilde{E}_{k,\omega}\, \partial\langle f\rangle/\partial v}{-i(\omega - kv)}}_{\text{coherent response}} + \underbrace{|e|\, \delta(x - x(t))\, \delta(v - v(t))}_{\text{discreteness source}}$$

$$\nabla^2 \phi = 4\pi n_0 |e| \int dv\, \delta f = 4\pi n_0 |e| \int dv\, f^c + 4\pi n_0 |e| \int dv\, \tilde{f}$$

$$\Rightarrow \quad \vec{\phi}_{k,\omega} = \frac{4\pi n_0 |e|}{k^2} \int dv\, \tilde{f} \Big/ \epsilon(k,\omega)$$

$$\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k} \int dv \frac{\partial \langle f \rangle / \partial v}{\omega - kv}$$

using u.p.o. :

$$\int dv\, \tilde{f}_k = \int dx\, e^{-ikx} |e|\, \delta(x - x(t))$$
$$= |e|\, e^{-ikvt}$$

so

$$\vec{\phi}_k(t) = \underline{\underline{\epsilon}}^{-1}(k,t) \frac{4\pi n_0 |e|}{k^2} e^{-ikvt}$$

Note: strictly speaking, have:

$$\underline{\underline{\epsilon}}(k,t)\, \vec{\phi}_k(t) = \frac{4\pi n_0 |e|}{k^2} e^{-ikvt}$$

$$\underline{\text{so}} \quad \phi_k(t) = \underbrace{\underline{\underline{\epsilon}}^{-1}(k,t) \frac{4\pi n_0 |e|}{k^2} e^{-ikvt}}_{\text{driven solution (discreteness)}} + \underbrace{\phi_k^{\text{homog.}} e^{-i\omega_k t}}_{\text{homogeneous solution}}$$

$$\omega_k = \omega_r(k) + i\omega_i(k) \Rightarrow \text{eigenmode freq}$$

Now, – time asymptotically ....

– for $\omega_i(k) < 0 \Rightarrow$ collective modes damped ...

$\Rightarrow$ only discreteness driven solutions persist ....

Catch: $\to$ For $\omega_i \lessapprox 0 \Rightarrow$ i) need wait quite a long time.

ii) for sufficient source strength, amplification to nonlinearity occurs ...

n.b. moving toward, but not to, marginal stability $\Rightarrow T_{relax} \to \infty$

$\to$ if unstable modes, require ultimate nonlinear damping to balance noise

i.e. $\epsilon_{IM} = \epsilon_{IM}(k, \omega, \langle \tilde{\phi}^2 \rangle)$ ...

"noise" = thermal + nonlinear, in that case

Proceeding, then test particle model $\Rightarrow$

$$\boxed{\langle \hat{\phi}^2 \rangle_{k,\omega} = \left(\frac{4\pi n_0 |e|}{k^2}\right)^2 \int dv_1 \int dv_2 \frac{\langle \tilde{f}(1)\tilde{f}(2)\rangle_{k,\omega}}{|\epsilon(k,\omega)|^2}}$$

$\therefore$ all content $\rightarrow \langle \tilde{f}^2 \rangle_{k,\omega}$ (abbreviation)
$\rightarrow \epsilon(k,\omega)$

Now, for discreteness noise:

$$\tilde{f} = \frac{1}{n} \sum_{i=1}^{N} \delta(x - x_i(t))\, \delta(v - v_i(t))$$

$\rightarrow$

$$\left. \begin{aligned} x_i(t) &= x_{i0} + v_i t \\ v_i(t) &= \text{const} \end{aligned} \right\} \text{u.p.o.}$$

$\rightarrow$ assume (discrete) uncorrelated test particles, so:

so $\langle \ \rangle = n \int d\underline{x}_0 \int d\underline{v}_i \, \langle f(v_i) \rangle$

$\underbrace{}_{\text{@ Maxwellian}}$

i.e. simple avg. over equilibrium distribution

$(k_B T \gg e^2/\bar{r})$

so

$$\langle \tilde{f}(1)\tilde{f}(2)\rangle = n\int dx_i \int dv_i \left[\left(\frac{1}{n}\sum_{i=1}^{N} \delta(\underline{x}_1 - \underline{x}_i(t))\,\delta(\underline{v}_1 - \underline{v}_i(t))\right) * \frac{1}{n}\sum_{j=1}^{N} \delta(\underline{x}_2 - \underline{x}_j(t))\,\delta(\underline{v}_2 - \underline{v}_j(t))\right] \langle f\rangle$$

$$= \frac{1}{n}\,\delta(\underline{x}_1 - \underline{x}_2)\,\delta(\underline{v}_1 - \underline{v}_2)\,\langle f\rangle$$

as avg. vanishes unless $\begin{pmatrix}\underline{x}_i\\ \underline{v}_i\end{pmatrix} = \begin{pmatrix}\underline{x}_j\\ \underline{v}_j\end{pmatrix}$

N.B. : Uncorrelated test particles can only correlate with themselves...

so

$$\boxed{\langle \tilde{f}(1)\tilde{f}(2)\rangle = \frac{\langle f\rangle}{n}\,\delta(\underline{x}_1 - \underline{x}_2)\,\delta(\underline{v}_1 - \underline{v}_2)}$$

Discreteness correlation

See Pg. 71 for furthur details ...

Details of TPM – Discreteness Correlation

1.) Discreteness Correlation Function

$$\tilde{f}(1) = \frac{1}{n} \sum_{i=1}^{N} \delta(x_1 - x_i(t))\, \delta(v_1 - v_i(t))$$

$$\langle \; \rangle = \int dx_i \int dv_i \; n \langle f \rangle$$

$$\begin{cases} 1 = x_1, v_1, t \\ 2 = x_2, v_2, t \end{cases}$$

$$\langle \; \rangle := \int dx_i \int dv_i\, n \langle f \rangle$$

$$\langle \tilde{f}(1) \tilde{f}(2) \rangle =$$

$$= \int dx_i \int dv_i \; n \left(\frac{1}{n} \sum_{i=1}^{N} \delta(\underline{x}_1 - \underline{x}_i(t))\, \delta(\underline{v}_1 - \underline{v}_i(t)) \right) \left(\frac{1}{n} * \sum_{j=1}^{N} \delta(\underline{v}_2 - \underline{v}_j(t))\, \delta(\underline{x}_2 - \underline{x}_j(t)) \right)$$

$$= \int dx_i \int dv_i \; \frac{\langle f \rangle}{n} \sum_{ij=1}^{N} \Big[\delta(\underline{x}_1 - \underline{x}_i(t))\, \delta(\underline{x}_2 - \underline{x}_j(t)) * \delta(\underline{v}_1 - \underline{v}_i(t))\, \delta(\underline{v}_2 - \underline{v}_j(t)) \Big]$$

only $\neq 0$ if arguments interchangeable

$$= \int dx_i \int dv_i \; \frac{\langle f \rangle}{n} \Big[\delta(\underline{x}_1 - \underline{x}_2)\, \delta(\underline{x}_i - \underline{x}_j)\, \delta(\underline{v}_1 - \underline{v}_2) * \delta(\underline{v}_i - \underline{v}_j) \Big]$$

$$= \frac{\langle f \rangle}{n}\, \delta(\underline{x}_1 - \underline{x}_2)\, \delta(\underline{v}_1 - \underline{v}_2)$$

[particles uncorrelated unless same]

- memory
- order

ergodic $\to$ ensemble, time

$$\langle \tilde f(1)\tilde f(2)\rangle = \langle \tilde f(0)\tilde f(2-1)\rangle$$

stat
homog.

$$\langle \tilde f(1)\tilde f(2)\rangle_k = \int e^{-ik(x_2-x_1)}\langle \tilde f(1)\tilde f(2)\rangle\, dx_-$$

2 'ahead' 1 $\Rightarrow$

$$\langle \tilde f(1)\tilde f(2)\rangle_{k\omega} = \int_0^\infty d\tau\, e^{i\omega\tau} U(2,\tau)\int e^{-ik(x_2-x_1)}\langle \tilde f(1)\tilde f(2)\rangle\, dx_-$$

(2) forward in time

$$+\int_0^\infty d\tau\, e^{i\omega\tau} U(1,-\tau)\int e^{-ik(x_2-x_1)}\langle \tilde f(1)\tilde f(2)\rangle\, dx$$

(1) backward

(2) $\Rightarrow$ $x_2 \to x_2 + v_2\tau$

$$(2) = \int_0^\infty d\tau \int e^{-ik(x_2-x_1)}\, e^{i(\omega-kv_2)\tau}\langle \tilde f(1)\tilde f(2)\rangle\, dx_-$$

$$= \int_0^\infty d\tau\, e^{i(\omega-kv_2)\tau}\langle \tilde f\tilde f\rangle_k$$

$x_1 \to x_1 - v\tau$

$$(1) = \int_0^\infty d\tau\, e^{i(\omega-kv_1)\tau}\langle \tilde f\tilde f\rangle_k$$

$$(2) = \frac{-1}{i(\omega-kv_2)}\langle \tilde f\tilde f\rangle_k$$

$$(1) = \frac{-1}{}\langle \tilde f\tilde f\rangle_k$$

$$\langle \tilde{f}\tilde{f} \rangle = \frac{1}{n} \langle f \rangle \delta(x_-)\delta(v_-)$$

$$\langle \tilde{f}\tilde{f} \rangle_u = \frac{1}{n} \langle f \rangle \delta(\cancel{v}_-) \quad \rightarrow \text{rec}$$

$v_1 = v_2$

$$① + ② = \left(\frac{i}{\omega - kv} + \frac{i}{\omega - kv} \right) \langle \tilde{f}\tilde{f} \rangle_u$$

$$\text{re}\,(① + ②) = 2\pi\,\delta(\omega - kv)\,\frac{\langle f \rangle}{n}\,\delta(v_-)$$